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## CRITICISMS AND DISCUSSIONS.

### ON THE LOGICAL INTRODUCTION TO THE THEORY OF FUNCTIONS.<sup>1</sup>

It seems as though it would still be premature to pronounce definite judgment on the results obtained by logisticians, to attempt immediately to draw up, as it were, its scientific balance-sheet. Many questions that come under the logistical doctrine have not yet been solved, and one of the most important works by Whitehead and Russell, a work which must contribute to the establishment of this branch of science, has been announced, though it has not yet been published.<sup>2</sup> Notwithstanding this, it may not be without interest to recapitulate certain points in the discussion which has recently been raised along these lines between mathematicians and logicians, and to attempt to deduce some conclusions therefrom.

This controversy had a metaphysical point of departure. The first thing was to find out what element of intuition and of logic there was in mathematical reasoning. Kant regards intuition (*Anschauung*) as a mode of thinking which involves sensibility; in his philosophy, the determination of the nature of this intuition is of supreme importance, since, in his system, analysis of the elements of knowledge is essential. The investigation, however, which aims at determining the precise characteristics of intuition and the relations of this intuition to thought in general, constitutes a transcendental speculation, which could not be the real object of debate, the theory of intuitionistic metaphysics being no more susceptible of proof than that of the intellectualistic metaphysics of logicians. In these matters, the argumentations of the most learned mathematician have no greater value than the discourse of the most ordinary philosopher,

<sup>1</sup> [Authorized translation by Fred Rothwell from "Sur l'Introduction logique à la théorie des fonctions," *Revue de Métaphysique et de Morale*, Vol. XV, 1907, pp. 186-216.]

<sup>2</sup> [Whitehead and Russell's *Principia Mathematica* began to be published in 1910 and is not yet complete.]

for the very reason that the mathematician no longer thinks as a geometrician but as a metaphysician. The problem, stated in scholastic terms, admits of no solution.

Nevertheless, what seemed to constitute the interesting element in this philosophical debate, was that the existence of a scientific theory, or at all events of a doctrine which offers a positive character, appeared to be connected with its solution. If intuitionism is the true doctrine, the works of Boole, Schröder, Frege, Peano, Whitehead and Russell must disappear; if logical intellectualism expresses true philosophy, the works of the logisticians are justified. Well then, in our opinion there is no solidarity at all between the metaphysical controversy, to which we have just alluded, and the question as to whether this special science called logistics has a solid foundation or not: the existence of logistics no more depends on controversies as to the nature of mind than the legitimacy of experimental physics is connected with discussions as to the inmost essence of matter. No doubt certain logisticians have attributed excessive importance to their own science, and, in a sense, have transformed it into scholasticism; this metascientific extension of a positive doctrine is illegitimate and raises insoluble antinomies. But we believe there to be a scientific and restricted conception of logistics, which is free from the fluctuations of philosophic thought: we will attempt to prove this theory shortly. Before entering into this essential point, we must remember that the constitution of logistics is in conformity with the meaning of the evolution of the sciences, and, without entering into many historical details, a few brief indications will enable us to justify this affirmation.

It is known that the works of Cauchy and Abel are at the root of the development of analysis in the nineteenth century, and that logical precision and rigorous demonstration characterize the work of these great geometricians. In 1826, Abel wrote to Hansteen: "I shall devote all in my power to bring light into the vast darkness now reigning in analysis. . . . In higher analysis, very few propositions are demonstrated with strict rigidity. Everywhere we find that unfortunate method of inferring from the special to the general. . . ."<sup>3</sup> This need of strict demonstration corresponded to logical requirements, to the necessity of reducing as far as possible the approximations of the imagination of inventors. Cauchy and Abel have contributed largely to the acceptance of the principle—now undisputed

<sup>3</sup> Abel, *Œuvres*, Sylow and Lies edition, II, 263.

—that no propositions in mathematics can be accepted unless it has been fully demonstrated. The geometricians of the eighteenth century, however, less exacting than modern mathematicians, were satisfied with methods the exactness of which left much to be desired, since for the most part they trusted to their intuition.<sup>4</sup> Suffice it if we recall the erroneous conception of divergent series, of Euler's series, for instance,

$$1-1+1-1\ldots$$

As the sum of this series is alternately 1 and 0, according as we take an odd or an even number of terms, the geometricians of the eighteenth century regarded it as natural to take the arithmetical mean of 1 and 0, i. e.,  $\frac{1}{2}$ , as the sum of the series. This reasoning, however, is inexact, "for it is easy to form series the terms of which depend on a variable, and which reduce to Euler's series for a particular value of this variable, in such a way that the limiting value of the series is absolutely any number whatsoever."<sup>5</sup> The geometricians of the eighteenth century employed divergent series because these series were successful empirically; they did not trouble about the logical difficulties which the series might raise. What we have just said regarding divergent series would be equally correct regarding many other branches of mathematics. Here again we would refer to the transformations in the definition of the integral. Analysts, from the time of Cauchy and Dirichlet, have endeavored to substitute, for the confused notion of the integral considered as an area, a rigorous analytical definition. Riemann's works throw light on the matter, whilst the results obtained by Henri Lebesgue at the present time must be mentioned. Here too, exact analytical definitions are substituted for vague intuition. We may therefore say, confining ourselves strictly to historical truth, that the analysis of the eighteenth century was far less rigorous than that of the nineteenth, and, unless we assert that the results of the nineteenth century are inferior to those of the eighteenth, we must acknowledge that the entire evolution of mathematical science points to progress along the lines of a greater logical precision.

The affirmation of Pierre Boutroux that "it is frequently advantageous, in the theory of functions, to substitute for the study

<sup>4</sup> We use the word "intuition" in its ordinary meaning. Intuition is a *rapprochement* made by the imagination, a *rapprochement* that is not tested by logical criteria.

<sup>5</sup> Borel, *Leçons sur les séries divergentes*, p. 6.

of a development that of less precise and more *intuitive* characters,"<sup>6</sup> must be understood as meaning that, in research, it is convenient to employ less rigorous methods; though unquestionably the results obtained, before being definitely incorporated in science, must be verified by the most exact methods. To assign any other meaning to this affirmation would be to deny the entire development of modern mathematics.

Ever since the progress of mathematical methods consisted in attributing an ever greater part of logical rigor and precision, we had to examine the very foundations of the reasonings of geometers, to analyze the fundamental types of demonstrations, and to establish the indefinable elements on which these demonstrations are based. Logistics, therefore, was in conformity with the spirit of the works of Cauchy and Abel. Still, though the historical development of mathematics certainly seems to prove that logistics is legitimate, *de jure*, at all events, how far can it be said that the logical reduction of mathematical reasonings is accomplished *de facto*? Do the nine indefinable and the twenty indemonstrable propositions of Russell constitute the whole logical framework of mathematics? These are questions to be examined in detail. It seems that there are problems which Russell has solved and others in which he has failed; it would scarcely be reasonable to expect him, unaided, to build up the whole of logistics. Note, moreover, that Russell would appear to be mistaken as to the import of logistics when he says: "The fact that all mathematics is symbolic logic is one of the greatest discoveries of our age."<sup>7</sup> It seems, however, that he himself restores logistics to its true import, when he regards it as a sort of speciality,<sup>8</sup> a branch of mathematics, unknown indeed to the majority of mathematicians, and not the indispensable instrument of all mathematical investigation. What are the exact limits of this new branch of mathematics? What is its positive function, and what are its relations to the great mathematical theories, especially the theory of functions? These are the questions that have to be answered. They are weighty problems, however, and we do not pretend to solve them in the following pages. We will confine ourselves to emphasizing certain points.

<sup>6</sup> Pierre Boutroux, *Thèse sur quelques propriétés des fonctions entières*, p. 1.

<sup>7</sup> Russell, *Principles of Mathematics*, 1903, p. 5.

<sup>8</sup> Russell, *Revue de Métaphysique et de Morale*, Sept. 1906, p. 628.

## I. THE GROUND COVERED BY LOGISTICS.

We will at the outset attempt to give a few indications as to the determination of the limits of the domain of logistics. It must have boundaries in the sense of metaphysical abstraction (a upper limit if you like), signifying that it must, by an explicitly formulated postulate, put a stop to all digressions in the barren domain of scholasticism.<sup>9</sup> It must likewise have boundaries as regards its applications (lower limits), or, if you will, its extension to the domain of positive sciences, in order to avoid clashing with existing methods. We shall take for our starting point Whitehead's definition of a calculus in general. A calculus is defined as "the art of manipulation of substitutive signs according to fixed rules, and of the deduction therefrom of true propositions."<sup>10</sup> According to this meaning, still a very vague one, a calculus may apply to two kinds of mathematical entities: *non-numerical* and *numerical*. These two kinds of entities differ in the fact that the former verify the following laws:

$$\begin{aligned} a &= aa \\ a &= a + a \\ a + ab &= a; \end{aligned}$$

whereas the latter satisfy the well-known rules:

$$\begin{aligned} aa &= a \\ a + a &= 2a \end{aligned}$$

(we give these rules only in order to state exactly what we think, we do not mean that the elementary laws governing the operations of ordinary arithmetic can be applied to all numerical mathematical entities).

The field of a calculus, comprising both numerical and non-numerical entities, contains all possible forms of thought: according to this general sense, then, logistics is a calculus.

But with regard to a calculus in general, there may first be asked the necessary philosophical question: what relation does reality hold to rules and to substitutive symbols? In our opinion, a problem thus stated has no scientific meaning, hence, whenever problems of this kind are asked, we may neglect them. In doing so, however, by a tacit postulate we confer absolute value on the logico-mathematical laws, we apply to them the idealistic principle (the laws of mind are

<sup>9</sup> Those who regard this precaution as unnecessary have only to read the articles on this subject in the *Revue de Philosophie*.

<sup>10</sup> Whitehead, *Universal Algebra*, p. 4.

the laws of things) which others apply to metaphysical principles and we thus put a stop to all wandering from the point in the field of scholasticism, as well as to questions which admit of no reply, such as the following: what is the objective value of symbolism? We should not have dwelt so long on this matter, had not Louis Couturat, an authority in questions of logic, affirmed that speculative philosophy comprises "in addition to the methodology of the sciences, the epistemology or criticism of the principles of the sciences, the general theory of knowledge, and finally metaphysics as the science of being, so far at least as this latter is known and knowable, and is conceived in its relations to the mind."<sup>11</sup> Now, if the methodology and the epistemology of the sciences are legitimate, it appears to us erroneous to affirm the justification of a theory of knowledge and of a metaphysic. Having already explained our position on this point, we will not dwell on it any longer.<sup>12</sup>

It remains for us to determine the lower limit of the field of application of logistics, a far more delicate task, for it does not appear as though we could yet obtain any definite results from the controversy on the subject. The problem is stated in the following general terms: apart from a certain domain proper to it, which Hadamard defines as that of the analysis of mathematical reasonings<sup>13</sup> and which we shall presently attempt to determine a little more explicitly, should logistics extend, as a new general method, to the different parts of mathematics? Is there a time when the logician must give way to the mathematician or can they both travel together? Is logistics, in every respect, a *substitute* for mathematics? If Peano has not fully adopted the latter alternative, he at least seems to regard logistics as a new algebra. "He uses it," says Jules Richard, "*as algebra is used* for stating propositions and deducing them from one another according to fixed rules. As I could prove by numerous quotations, he has never thought of making up for the primary notions of science, that of number, for instance, by this algebra."<sup>14</sup> But does not this mode of regarding logistics imply an illegitimate extension of its field of application? The argument that merely says that the *Formulaire* exists,<sup>15</sup> i. e., that it is in print, is of no great importance. The request to the opponents of logistics to find errors

<sup>11</sup> Couturat, *Revue de Métaphysique et de Morale*, May 1906, p. 340.

<sup>12</sup> *Revue de Métaphysique et de Morale*, July, 1905.

<sup>13</sup> Hadamard, *Revue générale des Sciences*, 30 Oct., 1906, p. 909.

<sup>14</sup> J. Richard, *Ibid.*, 30 Nov. 1916, p. 957.

<sup>15</sup> Couturat, *Revue de Métaphysique et de Morale*, March 1906, 220.

of demonstration in the *Formulaire*<sup>16</sup> is certainly more serious, though it does not remove the main objection. If logistics, when it takes up the work of the mathematician himself, is but a *translation* into other signs of ordinary mathematical reasoning, as it is accused of being, then it can make mistakes only if the original reasoning has made mistakes, in this case, logistics is blamed not for its falsity but for its *uselessness*: a method having justified its existence either when it permits of the solution of problems not previously solved, or when it is more precise and rapid than existing methods. The logistical method has not provided a solution of problems strictly mathematical and not previously solved, nor can it lay claim to rapidity; so that it remains to be seen whether, in the strictly mathematical domain, not only in the domain of the principles of mathematics, it possesses qualities of exactness which the standard methods do not possess, by explicitly formulating all the postulates which are mostly used unconsciously.

As a matter of fact, say the logisticians, the reason why the mathematician is not mistaken is because he is guided by unconscious logic. Let us see, then, in the most elementary of examples,—the arithmetical addition of integers,—if there is room for the intervention of logistics in the strictly scientific domain. We will leave out of account the questions on principles (definition of number, etc.), since, from our point of view, these come under the heading of logistics.

The operation  $+$  will be characterized, as in all treatises on arithmetic, as a primitive operation having the four properties:

- 1° Of being associative;
- 2° Of being commutative;
- 3° Of having zero for its modulus;
- 4° Of being such that, if to a number  $a$  we add numbers differing from one another, we still obtain numbers differing from one another.

Everyone knows that these properties were formulated long before the existence of logistics. Suppose the system of numeration determined, as well as the rules for writing numbers in the chosen system (we refer to treatises on arithmetic for details of three rules) and that we have to add together  $5+7+(3+4)=19$ . Will the rules established by arithmetic enable us to avoid all error? Are mistakes avoided by virtue of an unconscious logic or by virtue of the rules

<sup>16</sup> *Ibid.*, p. 221.

of arithmetic? In a word, does not arithmetic itself use logic? And is not all error in calculations of addition impossible when we correctly apply the rules of arithmetic? Is not the intervention of a logical calculation, in verifying the correctness of the operation, supererogatory?

We do not think there can be any doubt on the matter. If our example is regarded as too simple, the same question may be asked in the case of trigonometrical calculation. At the outset, we may introduce the trigonometrical functions sine  $x$  and cosine  $x$  in the way indicated by Jules Tannery so as to reduce experimental data as much as possible: "We also establish from geometrical considerations the formulas:

$$^a \begin{cases} \sin (a+b) = \sin a \cos b + \cos a \sin b. \\ \cos (a+b) = \cos a \cos b - \sin a \sin b \end{cases}$$

I will show (taking their existence for granted) how we may determine all the continuous functions  $\phi(x)$ ,  $\psi(x)$  which have properties defined by the formulae:

$$^b \begin{cases} \phi(a+b) = \phi(a)\phi(b) - \psi(a)\psi(b) \\ \psi(a+b) = \psi(a)\phi(b) + \psi(b)\phi(a) \end{cases}$$

and likewise satisfy another condition. . . . Designating by  $\cos x$  and  $\sin x$  functions which are known to be continuous and which must satisfy the equations (a) . . . we must have the relation

$$(\gamma) \sin^2 x + \cos^2 x = 1'' \quad 17$$

The functions  $\sin x$  and  $\cos x$  are determined by formula (a) and ( $\gamma$ ). Considering the functions  $\sin x$  and  $\cos x$  as given by their developments in series; it may be shown how these functions effectively satisfy the conditions with which we started. Without taking up the demonstrations in every detail, we see that here is a starting point for a rigorous establishment of the formula of elementary trigonometry.<sup>18</sup> By strictly keeping to the rules thus obtained, we may be certain that we shall never make mistakes in any particular trigonometrical calculation. It will not be unconscious logic, but a literal application of the formula that will have enabled us to avoid mistakes. Moreover, as all the reasonings of algebra, analysis and geometry are as rigorous as those just mentioned, and are all ob-

<sup>17</sup> J. Tannery, *Introduction à l'étude des fonctions d'une variable*, First edition, p. 146.

<sup>18</sup> *Ibid.*, p. 157.

tained by the ordinary rules of calculation, it is not clear what useful a part logistics is capable of playing within the particular domain of these different sciences.

Perhaps Peano and his collaborators had the impression that, by confining themselves to the determination of the grammatico-logical principles of mathematics, logistics would constitute no more than a limited speciality, and as the idea of a new general method was probably continually in their minds, they have included in the *Formulaire* a number of pages which, strictly speaking, ought not to be there. As an instance, amongst many others which might be mentioned, we will refer to pages 35, 40 and 41 in volume V of the *Formulario matematico* . . .

Speaking generally, we do not see what practical algebraical calculation is to gain from these evident remarks. From the beginning of algebraical calculation, strictly so called, *the rules of this calculation have become the logical rules*, and no further search is required. To teach the first elements of calculation, however, phrases have had to be employed; indeed, it would be impossible to explain the beginning of mathematics without a preliminary grammatical discourse. "The study of grammar," writes Russell, "is in my opinion capable of throwing far more light on philosophical questions than philosophers generally suppose."<sup>19</sup> Thus we meet with Peano's signs for implication, conjunction, belonging to a class, etc. These signs are substitutive for fundamental ideas. How are we to characterize these ideas, how employ them, and how do they relate to mathematics, properly so called? These are questions to which a reply must be given and which must be treated with the same precision as mathematics itself, if we would prevent the beginnings of arithmetic, analysis and the other branches of mathematics from being lost in mist and fog.

Russell has already thrown considerable light on these questions in Chapter V of the *Principles of Mathematics* which deals with denoting. He brings forward the part played by the words *all, every, any, a, some, the*.<sup>20</sup> As we cannot imagine the state of a man's mind from which all grammatico-logical notions were banished, the very principle of all positive philosophy compels us to study these notions as given facts, to set up the laws of the combinations proper to them, as well as the relations which unite them to mathematical science

<sup>19</sup> Russell, *Principles of Mathematics*, p. 42.

<sup>20</sup> There has since appeared an article by Russell in *Mind* (Oct. 1905) on the question of denoting.

strictly so called. Such is the positive field for the application of logistics, a field which constitutes the logical introduction to the theory of numbers and to the theory of functions.

If we agree with Hadamard<sup>21</sup> that the problem which consists in absolutely reducing the mathematical principle to the grammatico-logical one, or vice versa, constitutes a metaphysical problem that has no possible solution, there would still have to be set up positive and exact correspondences between the notions of both orders. Even admitting, though it is not so, that Aristotle formulated for all time the laws of logic, without the possibility of either adding to or removing anything from its formulas—the theory of Poincaré, which the illustrious mathematician himself destroys by acknowledging that the notion of propositional function constitutes a “happy invention,”<sup>22</sup>—we should still have to determine the correspondence between syllogistic and grammatical notions, on the one hand, and mathematical calculation, on the other. For instance, Whitehead has plainly shown the range and meaning of syllogistic by exhibiting it in its symbolical form; it becomes a particular instance of the general methods of elimination: “It is evident that each syllogism is simply a problem of elimination of the middle term.”<sup>23</sup>

The precise determination of the grammatico-logical elements which come into mathematics responds to that need for strictness which impelled Abel to take up again the demonstrations of the theorems of the higher analysis which he regarded as inadequate. This work, however, may also have another effect, that of improving the instrument of non-numerical logic, and of rationalizing grammatical forms by comparing them, and as far as possible reducing them, to mathematical forms.

Consequently, the analysis of mathematical reasonings, i. e., the determination of the grammatico-logical types they contain, constitutes the distinctive domain of logistics. It must be admitted that the determination of the nature and range of the principle of complete induction, for instance, should be dealt with logistically, and that no other way of attacking the problem is scientific. In metaphysical terms, Poincaré states the following problem: How, by an analytical method (conceived as based solely on the principle of identity), can *new* truths be discovered?<sup>24</sup> Poincaré finds the solu-

<sup>21</sup> Hadamard, *Revue générale des Sciences*, 30 Oct. 1906, p. 908.

<sup>22</sup> Poincaré, *Revue de Métaphysique et de Morale*, Nov. 1905, p. 827.

<sup>23</sup> Whitehead, *Universal Algebra*, p. 103.

<sup>24</sup> Poincaré *La Science et l'Hypothèse*, p. 10.

tion of the question in complete induction. Complete induction, then, which solves a philosophical problem, is thereby promoted to the dignity of a method-type for mathematics. It is possible, however, that the metaphysical problem originally stated may not be legitimate, scientifically speaking at all events. First, is it true that the analytical method is reduced to the application of the principle of identity? Couturat has refuted this purely tautological conception of the analytical method. Moreover, does the novelty of the discovery for which Poincaré wishes to account, constitute a psychological manifestation with reference to man, or does it correspond objectively to something? If we reveal the laws of nature, that means that they are new *to us*, just as America was new to Christopher Columbus in spite of the previous existence for centuries of the American continent. And is it not manifestly contradictory to think of attributing an absolute logical foundation to what is only psychological and human? Without advancing further into the scholastic labyrinth, these remarks suffice to show that the terms of the problem were badly determined: consequently, any solution of them will be disputable. Now, it is because complete induction solved a metaphysical problem in the mind of Poincaré that he has attributed to it a universal value which it cannot have. Subsequently, Poincaré mitigated the too absolute element in his original conception: "I saw in it the mathematical reasoning *par excellence*. I did not mean, as has been thought, that all mathematical reasonings could be reduced to an application of this principle."<sup>25</sup> But is it a fact that this is mathematical reasoning *par excellence*? As Poincaré's opponents, the logisticians, did not start with a general philosophical problem, and it must be recognized, that their point of departure, at all events, is in conformity with the positive method: they have studied in succession all the forms of mathematical reasoning, and, analysing the first logical notions, have found the principle of complete induction in the theory of *finite* numbers;<sup>26</sup> they would appear to have quite recognized the part this principle plays, regarding it as a fundamental element in the definition of these numbers. Now, are there other forms of demonstration foreign to the principle of induction (as, for instance, in the syllogism  $a=b$ ,  $b=c$ , therefore  $a=c$ ) and which are in no way applications of this principle? On this point, logistics has determined a certain number of types of reason-

<sup>25</sup> Poincaré, *Revue de Métaphysique et de Morale*, Nov. 1905, p. 818.

<sup>26</sup> Couturat, *Les principes des mathématiques*, p. 62.

ings (the principle of composition, of simplification, of contraposition, of exportation, of importation, of deduction, of substitution, principles of the logic of relations. . . .) the inanity of which would have to be shown before we could affirm the supremacy of the principle of complete induction. Moreover, it is clear that, if we give the name of complete induction to the faculty possessed by the human mind of forming general judgments, valid for an infinite number of cases, such a principle intervenes in every act of thought. There is nothing legitimate about this denomination however. It would appear that the method of complete induction strictly speaking, must only be used in its own proper domain (the theory of finite numbers) and in the cases related more or less directly to this domain. It appears, then, that the logisticians must finally win as regards the principle of complete induction, the problem being distinctly within their province, since we have to analyze a mathematical reasoning.

To sum up, however sarcastically logistics be regarded, it is a positive domain of investigation, an ensemble of questions to which answers must be given. Still, while we cannot ignore this branch of science, it would be equally erroneous to imagine that the essential of mathematical thought has been absorbed in the general theories of logistics and to think that all the other branches of mathematics are reduced to an automatic application of the rules laid down in the logical introduction. Even admitting, as we do, the justification of logistics, though circumscribing its domain very precisely, the absolute autonomy of mathematical thought must be acknowledged. What we mean is that, even if the general questions of logic were solved to the satisfaction of all, the difficulty of the real problems, which we find in pure mathematics or which the physicist asks the calculator, would be in no way lessened thereby. To parody a famous motto, we might say: Logic is well established, now begins the era of scientific difficulties.

In the two following paragraphs we will endeavor to set forth a few examples of problems to the solution of which logistics brings no light or help of any kind; thus we shall determine more positively what we have called its lower limit.

## II. ON THE DEFINITION OF IRRATIONAL NUMBER AND ON THE GENERALIZATION OF NUMBER.

In one article, we could not dream of examining in succession all the theories of logistics. We will content ourselves with taking, as an example, the definition of irrational number set forth in

chapter XXXIV of Russell's *Principles of Mathematics*. We will sum up what Couturat says of Russell's results: "His definition consists in identifying the irrational number with the *lower class* which previously served to define it. . . . We will give the name of *segment* to every not null class of rational numbers, which does not comprise all rational numbers, which comprises all rational numbers smaller than any one of its elements, and such that each one of its elements is smaller than another of its elements."<sup>27</sup> These last two conditions may be expressed in symbols. It is shown that there are more segments than rational numbers. "These segments, by definition, will be the real numbers."<sup>28</sup> The irrational numbers has been identified with the lower class.<sup>29</sup> Does this laboriously obtained definition, after criticism of the many definitions of the irrational number, throw the faintest light on the strictly mathematical difficulties which come under the theory of these numbers? By no means, as we shall at once show.

How far have the preceding definition and the different logistical theories dealing with the irrational numbers helped to solve the following problem?

A number being defined by an infinite succession of integers, to find whether this number is commensurable or incommensurable.<sup>30</sup> True, it is known that if a number is defined by its expansion in decimals, for the number to be commensurable it is necessary and sufficient that the series should be periodic from a certain number onwards. It is also known that if the number is developed in a continued fraction, in order that it may be commensurable the development must be limited. These rules, however, do not solve the general problem.<sup>31</sup> This is not all, the logistical theory does not permit us to recognize whether a definite incommensurable number is algebraic or transcendental, and in case it is algebraic to determine its degree.

As regards more especially transcendental numbers, their existence is by no means evident *a priori*, it results, as it known, from

<sup>27</sup> Couturat, *Les Principes des Mathématiques*, p. 85.

<sup>28</sup> *Ibid.*, p. 86.

<sup>29</sup> We might also identify the irrational number with the higher class.

<sup>30</sup> We shall use without distinction the terms "incommensurable" and "irrational," though both alike are defective. As a rule, the term "incommensurable" is used for dimensions.

<sup>31</sup> E. Cahen, *Eléments de la théorie des nombres*, p. 172 and p. 183.

a theorem of Liouville:  $p/q$  being an irreducible fraction, value approaching  $\xi$ , if we have

$$\text{Mod. } (p/q - \xi) < 1/q^{n+1}, \text{ since}$$

the values of  $q$  transcend all limit,  $\xi$  cannot be an algebraic number of the degree  $n$ . If we can prove this for each value of  $n$ ,  $\xi$  is transcendental.<sup>32</sup> Thus the existence of  $\xi$  demands a particular demonstration. We may add that, as regards algebraical numbers, there is a theorem of Lagrange which enables us to know when they are of the second degree. Any reader desirous of studying these questions more deeply will find important results in Edmond Maillet's work on the theory of transcendental numbers.<sup>33</sup> Especially do we draw attention to the theorem which supplies the necessary and sufficient condition for an irrational, positive, real number to be a transcendental number of Liouville.<sup>34</sup> In these investigations no mention is ever made of the formula of logistics. In a word—and it is all we wish to remember of the foregoing explanations—the logistical definition of the irrational and the considerations that logisticians may have advanced regarding it, whether put in formula or not, throw no light whatsoever on the determination of the distinctive characters of the commensurable and incommensurable numbers, of the algebraic incommensurables of the different degrees and finally of the transcendental numbers. In these matters, the results have been obtained by attacking the difficulties directly by the ordinary mathematical methods. Along these lines, we may add that there is much to be done.

It would be fastidious to mention all the main problems not solved by the ordinary methods, and for which logistics gives no help at all to the mathematician. As an example, we will cite the following case found in analysis. "It is known that two functions, which always have the same derivative, differ only by a constant, when this derivative is *finite*; as regards the general case, nothing is known"<sup>35</sup> or, along a totally different order of ideas, we will recall one of Fermat's enunciations, unsolved in the general case: the equation  $x^n + y^n = z^n$  is not resolvable into integers for  $n < 2$ . We must not

<sup>32</sup> Borel, *Leçons sur la théorie des fonctions*, p. 26.

<sup>33</sup> Maillet, *Introduction à la théorie des nombres transcendants*, 1906.

<sup>34</sup> *Ibid.*, p. 124.

<sup>35</sup> Lebesgue, *Leçons sur l'intégration et la recherche des fonctions primitives*, p. 75.

<sup>36</sup> Couturat, *Les principes des mathématiques*, p. 79.

forget, however, that we are writing in a philosophical review, and so we would rather examine a question of a more general character: the generalization of number will enable us to make a few interesting remarks. "The same logical method," writes Couturat, "enables us to explain the generalization of number. First, the rational numbers will be considered as relations between integers. . . . The positive and negative numbers also will be conceived of as logical theory of irrational numbers requires the previous study of the continuous, just as the logical theory of complex numbers presupposes the theory of space."<sup>37</sup> Logistics finds no serious difficulty, so long as it does not more than justify and organize the numbers, or, more generally, the mathematical entities which positive science has adopted. When we are dealing with new entities, however, the scientific use of which is still disputed, it does not appear as though logistics could afford any very effective help in the solution of the questions, so true is it that the introduction of the new entity is mainly justified by the mathematical use to which it is put. First and foremost, logic has no general process of generation for all possible entities—moreover, it seems as though experiment (for instance, the measurement of certain physical dimensions) plays an important part, at least as regards the origin of the new entities. Consequently, whenever a new entity obtrudes upon the attention of scientists, the question that is first asked is whether it is reducible to the known entities, and only special study in mathematics can afford us profitable information on this point. If logistics possessed a restrictive general principle (or a small number of such principles) which the new entities must verify, it would have a simple logical process which would permit of some of them being eliminated. It was thought that Hankel's principle of the permanence of the rules of calculation had this property; but, by a strange irony, it was the logisticians who pointed out that this was not so. This principle, remember, may be formulated as follows: "If, for the purposes of generalization, we can give up some particular property of an operation, we must on the other hand take care not to add any new property to those already used in the restricted operation, so that any rule set up for the generalized operation may also be applicable to the restricted operation."<sup>38</sup> To say that this principle is false means, in a word, that it is not possible to fix for the arithmetical operations general characters valid for all classes of numbers, and more generally for all classes of

<sup>37</sup> *Ibid.*, p. 81.

<sup>38</sup> Houël, *Cours de calcul infinitésimal*, I, 5.

mathematical entities, since each particular class of entities has its own distinctive operative laws. But then, such reasoning as that of Weierstrass, for showing that any hyper-complex number of an ensemble with  $n$  units "is decomposable into  $r$  complex numbers belonging respectively to  $r$  partial ensembles with *one* or *two* units . . . the ensembles with one unit being analogous to the ensemble of the real numbers . . . the ensembles with two units being analogous to the ensemble of the ordinary complex numbers,"<sup>39</sup> a reasoning that is possible only on condition that we suppose the associativeness, the commutativeness, and the distributiveness of the multiplication of the new complexes, is still quite formally correct, though it loses all general philosophic bearing since Hankel's principle is false. The field, then, remains absolutely open to the creation of new entities which will individually have to be subjected to a special mathematical investigation. We may add that the relations of the entities with their operative laws have nothing fixed: certain numbers, such as integers, are reproduced by addition and multiplication but others (algebraic numbers of the second degree, for instance), generate numbers of a denomination different from themselves by addition and multiplication.

Of all the new theories, however, the theory of transfinite numbers is by far the most instructive to examine; for the reason that it contains a class of new entities on the bearing of which mathematicians are not agreed. What are the transfinite numbers which must be admitted? Those of the first two classes? Those of a higher class? Are not the *ordinal* types called to a greater future than the *transfinite cardinals*? Must we reject all these numbers *en bloc*? Mathematicians are not agreed, and logisticians, hitherto, have come to no positive result. Still, if we trust to what the history of mathematics teaches us, there is every ground for believing that geometricians, of themselves, will solve the difficult problems contained in the new doctrines. Indeed, the theory of imaginaries and hyper-complexes had given rise to logical or philosophical difficulties over which mathematical thought, without the help of foreign allies, has succeeded in triumphing. In all probability it will be the same with the transfinite numbers. Here we do not claim to go to the root of this problem; we simply state at the outset that the criterion of mathematical utility constitutes, provisionally at least, a principle of elimination which will probably cause many parts of Cantor's work

<sup>39</sup> Couturat, *De l'Infini mathématique*, note I, p. 587.

to be condemned. This criterion has been formulated by Picard in the following terms: "There will be occasion to develop it (the arithmetic of the transfinite numbers) only if these views prove to be fruitful in analysis; consideration of the transfinite numbers has already enabled us to discover certain theorems, though we must say that they might have been obtained by another method."<sup>40</sup> Nor must we forget to make a distinction—one which might escape the notice of philosophers—between the theory of *transfinite cardinals* and that of *transfinite ordinals*. As regards the first theory, the opinion of Borel would seem provisionally to sum up the question best: "The second principle of formation cannot make us acquire the notion of a power we had not already, and it appears doubtful whether we have any definite idea of what a power beyond the second can be. . . . It cannot, however, be denied that at the present time the expression *transfinitely* has still a less precise meaning than the expression *indefinitely*, the result being that our precise knowledge of the different powers goes scarcely beyond the following remark: there are numerable ensembles and non-numerable ensembles, the later notion being mainly negative."<sup>41</sup> The theory of the ordinal types seems called upon to play a more important part than that of the transfinite cardinals. In this connection, let us consider what led Baire to introduce the notion of transfinite ordinal: "In the same way, for well ordered ensembles, we think it may be advantageous, when fixing the notion of *rank* in such an ensemble, to attach a new term to this notion and proceed as though all the elements of a well ordered ensemble had ranks determined once for all. Now, we have seen that integers were inadequate for this purpose, we shall give them new signs which will be the transfinite numbers."<sup>42</sup> To sum up, it would seem that the determination of the definite balance of the theory of transfinities comes within the province of the mathematicians, and that certain results only of Cantor's theory, but not all, as Russell seems to think, may be retained. Moreover, the existence of logistics, from the above explanations, is in no way connected with the success of the whole of transfinitism.

<sup>40</sup> Picard, *La science moderne et son état actuel*, p. 56.

<sup>41</sup> Borel, *Leçons sur la théorie des fonctions*, p. 122.

<sup>42</sup> Baire, *Leçons sur les fonctions discontinues*, p. 41.

### III. IRREDUCIBLE INTEGRALS AND THE NUMBER OF PRIMITIVE IDEAS.

We know that, in problems of quadrature, we are led to consider functions that cannot be integrated by means of algebraical expressions and elementary transcendentals (logarithm, exponential, circular functions) in finite number. Thus, we establish the existence of types irreducible to simpler forms. Then we say that there are new transcendentals. We cannot insist too strongly on the way in which mathematicians study these new entities.<sup>43</sup> The process we are about to characterize in a general way is of considerable philosophical importance, since the so-called chain of purely analytical deductions from mathematical propositions would seem at a given moment to be broken; supposing we are to imagine mathematical propositions as forming a chain. At a certain moment, we find mathematical entities which cannot be formally integrated by means of elementary forms in finite number. We do not on this account banish them from the field of analysis, but we study them as new objects with properties *sui generis*. They will have formulas of addition, multiplication, developments in series, etc., which are proper to them.

Take, for instance, the elliptic integrals which play a large part in analysis, and the question of the reducibility or irreducibility of differential equations. . . .

These irreducible mathematical entities constituting a non-enumerable infinity, how are we to reconcile this fact with the logical theory according to which there is only a finite number of elementary ideas? Are we to say that these original mathematical entities, transcendental numbers, irreducible integrals, form a *progression*? The term progression, however, in the logistical theory, signifies a sequence similar to the sequence of the natural numbers. A finite number of indefinables and of indemonstrable principles suffices to constitute the unlimited sequence of the natural numbers and their elementary properties. (Of course, the logistical doctrine throws no light on the inner nature of these numbers, as we may see by studying the theory of prime numbers.) As regards the natural numbers, the elementary operative rules are permanent for the entire sequence. However large a natural number, even if it extends to a hundred digits, the laws of addition, subtraction, etc., will remain the same

<sup>43</sup> To justify our conception there might be found other instances than the one we are examining. This, however, has the advantage of emphasizing the fundamental notion of irreducibility.

for it as for the prime numbers, unless we contest the value of the principle of complete induction. The ensemble, however, of the irreducible transcendentals cannot be regarded as forming a progression. First, the operative rules are no longer retained; for instance, there is a formula of addition for the elliptic functions which is not the same (although these formulas have analogies) as the formula of addition of the trigonometrical functions. Again, there is no *consecutiveness* between the elements of such an ensemble, such as there is in the sequence of the natural numbers. Finally, the power of this ensemble is not like that of the sequence of the natural numbers, the power of the ensemble of the irreducible mathematical entities constituting a non-numerable infinity. On the other hand, however, we cannot think with an infinity of logical principles. No doubt such notions as those represented in mathematics by the signs of an infinite sum, an infinite product, an integral, imply an infinity of elements, but as regards these notions we must agree with Russell and Couturat that they are thought in comprehension, not in extension. It must then be concluded that we think with a finite number of logical constants (indefinable notions, indemonstrable propositions) which enable us to study the transinitely infinite universe of mathematical entities. We will sum up what we think regarding the number of the elementary ideas by saying that the system of our logical ideas constitutes a closed system, and that the universe of abstract and concrete entities to which it applies will always be for us an open system. But, for that very reason, the study of the universe of entities will never be completed, and it will force us to modify the closed system of the logical constants. We shall indeed be led to consider new irreducible entities, which will not necessarily come within our logical schemes; hence the need to perfect them. At a given moment of evolution, we always think with a finite system of logical constants; but one system succeeds another. Couturat has claimed the right, quite a legitimate one, to transform and improve academical logic, and has successfully refuted the extraordinary theory according to which logic alone, of all productions of the human mind, having its origin in the brain of Aristotle, constitutes an eternal and consequently divine monument. Just because logic, however, is subject to the general law of evolution, we cannot affirm that the logic of Russell, with its nine indefinables and its twenty undemonstrable propositions, possesses that character of perennality which we rightly refuse to attribute to the work of Aristotle.

Observe that evolution may take place by way of adjunction—as is the case with mathematics. It is not necessary, as is erroneously imagined, that it should take place by way of contradiction (Hegelian theory). The multiplication table, the formulas of trigonometry, said to have been first formulated by Hipparchus and, dating back several thousands of years, have never been contradicted, never ceased to be true.

#### IV. THE NOTION OF FUNCTION AND THE RESTRICTIVE CONDITIONS FOR ITS USE IN MATHEMATICS.

In the last two paragraphs we have indicated mathematical theories which are opposed to an illegitimate extension of logistics outside of its field of application. Inversely, we must not restrict this field too much. It would appear as though Pierre Boutroux has gone too far along this line and refuses to logistics the exercise of a legitimate right. In the determination of the conditions that restrict the notion of function, he thinks he has found an extra-logical operation,<sup>44</sup> consequently, a limitation to the domain of logistics. Later on we will examine more thoroughly this complex problem which raises numerous difficulties. In this short paragraph we will simply submit to the philosophers certain general results, which will enable us to make a few reservations as to the judgment of Pierre Boutroux.

It may be said that the most general notion of function is a discontinuous and nonuniform function. It is known, however, that analysis does not possess a theory of functions corresponding to this absolutely general type. Indeed, it is easy to show that we cannot at present set up a theory of the most general discontinuous functions. Let us follow Borel's reasoning: a continuous function may be given by means of a numerable infinity of conditions; it is not so with a discontinuous function in its most general meaning: such a function is defined by a *non-numerable* infinity of conditions; in practice, this is equivalent to saying that it is impossible to define it."<sup>45</sup> Consequently, we cannot think of approaching the theory of functions by allowing this notion to retain its greatest generality. Restrictions must be imposed on it.

Keeping to the notion of function, Lebesgue has retraced the history of the early transformations of this notion. We will briefly sum up his statement. At the time of Newton and Leibnitz, a

<sup>44</sup> P. Boutroux, *Revue de Métaphysique et de Morale*, July 1905, p. 633.

<sup>45</sup> Borel, *Leçons sur la théorie des fonctions*, p. 126.

function was generally called "a quantity  $y$  connected with a variable  $x$  by an equation, with a certain number of symbols of operations intervening (arithmetical, trigonometrical, logarithmic operations)."<sup>46</sup> Then, following on the problems of integration, we have been led to consider the case in which there is a geometrical relation between a function  $S(x)$  (*area*) and  $x$ . Afterwards, a distinction was made between "the geometrical figures defined with the aid of laws capable of being expressed by geometrical equalities and the figures that were not thus defined."<sup>47</sup> To the curves included in the first case corresponded the continuous functions (Eulerian continuity), to the curves of the second kind corresponded arbitrary functions which were not true functions." The continuous functions were the true functions."<sup>48</sup> Later on, Fourier overthrew this method of considering the functions, by showing that the "trigonometrical series which could be employed, in many cases, for the representation of continuous functions, might also be used for the representation of non-continuous functions formed of parts of functions."<sup>49</sup> Then Cauchy gives the following definition of function: " $y$  is function of  $x$  when to each state of magnitude of  $x$  there corresponds a perfectly determined state of magnitude of  $y$ ."<sup>50</sup> To Cauchy's mind, it was necessary that these correspondences should be analytical; later on, this restriction was removed. Without dwelling upon the important investigations of Riemann and the subsequent studies which have thrown light on this matter, we may mention summarily the results obtained by Baire. Baire asked himself what were the conditions which the discontinuous functions must satisfy to be representable by series of continuous functions. He first shows "that a function presenting a *finite number* of discontinuities is the limit of continuous functions."<sup>51</sup> In a more general way, he distinguishes the *punctually and totally discontinuous* functions, according as their oscillation has its minimum everywhere nil, or not.<sup>52</sup> (We know that the oscillation is the difference between the maximum and minimum values of the function in an interval.) He obtains the following theorem: "Every

<sup>46</sup> Lebesgue, *Leçons sur l'intégration et la recherche des fonctions primitives*, p. 1.

<sup>47</sup> *Ibid.*, p. 3.

<sup>48</sup> *Ibid.*, p. 3.

<sup>49</sup> Lebesgue, *Leçons sur l'intégration et la recherche des fonctions primitives*, p. 3.

<sup>50</sup> *Ibid.*, p. 4.

<sup>51</sup> Baire, *Leçons sur les fonctions discontinues*, p. 11.

<sup>52</sup> *Ibid.*, p. 75.

limit of continuous functions, is a punctually discontinuous function."<sup>53</sup>

After this too brief statement, we may say that the problem set by Brouwer is intimately connected with the history of the notion of function. Now, this history has never been specially and fully set forth. Consequently, so long as the equivalent of the historical work of Charles on the methods of geometry does not exist for the theory of functions, we cannot clearly deduce the part of real logic implied in the fundamental notion of function and in the conditions imposed on it. When Pierre Brouwer affirms that "the conditions by means of which we determine the idea of function have an indeterminate character, that they are not the known but the unknown quantity of a problem,"<sup>54</sup> or even that investigation into the conditions to be imposed on the functions constitutes an extra-logical operation, he adopts the point of view of discovery. First, however, the notion of function is with difficulty distinguished from that of *relation*, which seems to form part of those grammatico-logical notions mentioned at the outset of this study as forming part of the domain of logistics. Do not the most general conditions, at least, which we impose on the idea of function, also share to some extent in this logical character? We have just indicated the method which, by unfolding the historical origin and the developments of each notion, would in our opinion permit of a reply to this question; we cannot here develop it further.

#### V. CONCLUSION.

We will briefly sum up the conclusions drawn from this study.

Note first that the controversy, to which we alluded at the start, ceased to deal with the essential philosophical problems when it degenerated into an examination of the cases of mathematical *teratology*. The Burali-Forti antinomy, the Richard contradiction, the "All Cretans are liars" sophism are the extreme confines of the science. We do not deny that they are interesting, but there are more fundamental problems lying at the very heart of mathematical theories. To resolve to reduce the philosophy of mathematics to the solution of sophisms would be as unjust as to look upon pathological physiology as the model of all physiology. Now, the readers might have this impression on reading the latest discussions dealing with

<sup>53</sup> *Ibid.*, p. 83.

<sup>54</sup> P. Brouwer, *Revue de Métaphysique et de Morale*, July 1905, p. 628.

these questions. We may even go further and say that the elaboration of logistics does not involve all the difficulties which deserve to create reflection on the part of the critic of the sciences, and that there are real mathematical problems which give rise to philosophical questions of wide range.

Afterwards, we saw that logistics constitutes a restricted though indispensable branch of general mathematics.<sup>55</sup> Remember that Auguste Comte, who is usually quoted as an opponent of formal logic, conceived the possibility of this about the year 1830: "I do not know whether subsequently it will become possible to give *a priori* a genuine course of method altogether independent of the philosophical study of the sciences, but I am convinced that this cannot be done at present, the great logical processes not yet being capable of explanation with due precision, separately from their applications. I also make bold to add that, even if such an attempt could subsequently be realized, which indeed may be regarded as a possibility, this would still only be effected by the study of the regular applications of the scientific processes. . . ."<sup>56</sup>

The legitimacy of logistics, in its own sphere, can no longer be seriously denied; the only thing remaining is for us to find out how far, *in reality*, it has fulfilled its task. We may repeat, however, that it cannot claim the rôle of a new method of investigation; to compare logistics with the differential and integral calculus, for instance, would be altogether erroneous. From its beginning, the differential and integral calculus has afforded a solution of mathematical problems: problems of the quadrature and the tangent, more special problems such as that of the catenary. There is nothing of this in logistics, which remains the mathematics of the elements. Nor could it be regarded as a sort of universal speciousness, wherein mathematical thought would be wholly done away with.

To sum up, there are two conceptions of logistics: the one which assigns to it a special positive function, the other which, desiring to take up in it the whole of human thought, transforms it into an algebraical scholasticism as barren as that of the Middle Ages. As

<sup>55</sup> Peirce maintained that logic is a branch of mathematics, M. Couturat claims that mathematics is a branch of logic (a branch far more important than the trunk from which it sprang). The question thus formally stated appears to be of no great practical or scientific interest. The main thing is that there is a branch of positive studies called logistics, that this is studied in the form of a symbolical calculation, a calculation whose efficacy is *special* since it cannot solve problems in other branches of mathematics.

<sup>56</sup> Auguste Comte, *Cours de philosophie positive*, 1830 edition, I, 39.

we have seen, scientific thought comes up against real problems, facts in the solution or explanation of which the reduction of the ordinary mathematical methods to the logistical methods afforded no advantage whatsoever. Logistics, *qua* universal explanation, should be condemned by the same right as any kind of metaphysics, because, being practised outside of its field of application, it remains a mere sport of the mind, of no scientific use whatsoever. Indeed it is one thing to determine and to classify the grammatico-logical elements of ordinary language, which form an element in mathematics; it is another thing to claim absolutely to reduce all mathematics to these elementary forms. In the former case you act as a scientist, in the latter as a metaphysician. The complete identification of mathematical thought with the grammatico-logical elements which condition it is an illusion analogous to that of materialistic dogmatism, which wholly assimilates thought to the elements of the brain, its material conditions of production. The proof that this identification is ineffectual results from the fact that, if logistics made the mechanism of mathematical thought absolutely transparent, and if it practically absorbed this thought, we should find no more mathematical problems offering difficulties, which logistics would not be capable of solving immediately. Now, we know that nothing of the kind takes place.

In reality, we are confronted with an ensemble or irreducible methods and data which form the various branches of mathematics. In no way could we explain all this knowledge by a systematic philosophical view. At all events, such an explanation would have no objective value scientifically. We may, however, bear witness that scientific methods are organized and transformed according to the ideal of the logician, and that, in theoretical sciences, mathematical demonstration surpasses all other forms of thought (intuition, imagination, etc.). By studying the inner mechanism of this demonstration, we glimpse the logical meaning of "the mysterious unity manifested in analytical works apparently most remote,"<sup>57</sup> a unity the complete realization of which could not be regarded as effectively brought about, but only as ideally conceived, at the present stage of our knowledge.

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<sup>57</sup> Hermite, *Comptes rendus de l'Académie des Sciences*, 1862, Vol. LV, p. 91.